Equivalence of Faddeev-Jackiw and Dirac approaches for gauge theories

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Abstract

The equivalence between the Dirac method and Faddeev-Jackiw analysis for gauge theories is proved. In particular we trace out, in a stage by stage procedure, the standard classification of first and second class constraints of Dirac's method in the F-J approach. We also find that the Darboux transformation implied in the F-J reduction process can be viewed as a canonical transformation in Dirac approach. Unlike Dirac's method the F-J analysis is a classical reduction procedure, then the quantization can be achieved only in the framework of reduce and then quantize approach with all the know problems that this type of procedures presents. Finally we illustrate the equivalence by means of a particular example.

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1   Introduction

The classical treatment of the dynamics of gauge theories (including those with reparametrization invariance) described by a variational principle was first solved by Dirac [1] and Bergmann [2] in the early 50's. Dirac was mainly interested in the Hamiltonian approach to general relativity, and his leading efforts in this field were completed in the 60's with the ADM formalism [3]. The application of Dirac's work to gauge theories like EM or YM came later in the 70's (for instance, the basic canonical commutations for the electromagnetic potential in Coulomb gauge, first obtained through heuristic methods, were then understood [4] as a direct application of the Dirac bracket). Also in the 70's the geometrization of Dirac's method was successfully addressed [5] and in the 80's there appeared general proofs of the equivalence of the (classical) Hamiltonian and Lagrangian treatment for gauge theories [6]. With regard to the quantization program, Dirac's approach opened the way to the so called Dirac method of quantization, where constraints are implemented as operators in Hilbert space. The equivalence of this method with the so called *reduced* quantization (i.e., when the classical degrees of freedom are eliminated before quantization) is still a subject of controversy [7]. Let us mention also the discovery in the 70's of the BRST symmetry [8], which appeared as a reminiscence of the classical gauge symmetry after fixing the gauge through Faddeev-Popov procedure [9]. In this sense, the classical approach by Dirac had its quantum continuation through the BRST methods. These methods have provided the most powerful tool, the field-antifield formalism, to quantize any kind (reducible gauge algebra, soft algebra, open algebra, etc.) of local gauge field theory [10].

The main features of Dirac approach, either in Lagrangian or Hamiltonian formalism are a) the possibility to keep all the variables in phase space or velocity space, b) the construction of an algorithm to determine, through a step by step procedure, the final constraint surface where the motion takes place, c) the elucidation of the true degrees of freedom of the theory, separated from the gauge –and hence unphysical– ones. In general, quotienting out the gauge degrees of freedom can be done by "fixing the gauge", which amounts to the introduction of a new set of constraints from the outset, the Gauge Fixing constraints.

More recently, a new method to classically deal with gauge theories was devised through the work by Faddeev and Jackiw[11]. Unlike Dirac's, in the F-J method, the variables are reduced to the physical ones and, at least formally, the procedure looks simpler. We have considered that it is worth to explore F-J method to see to what extent it differs –if it does– from Dirac's, and what are the advantages of each one. Let us immediately state our conclusion: though technically different, both methods are equivalent, as it should be. The main difference in procedure is that F-J method is a method of *reduction* to the physical degrees of freedom –and its variational principle.

It is true nevertheless that Dirac's method also includes the possibility to eliminate variables. In fact, the adoption of the Dirac's bracket (which is the bracket associated to the symplectic form that one can define in the second class constraint surface as the projection of the symplectic form in phase space) allows for the elimination of a variable for every second class constraint. Instead, in the F-J procedure, even in the case when

some first class constraints are present, the reduction can be still performed, although the projected symplectic form becomes degenerate. This implies that some dynamical variables and its equations of motion may be lost. A key feature of F-J procedure is that this fact does not affect the physical content of the theory.

As we will see in the next section, F-J procedure focuses exclusively getting the equations of motion (or, equivalently, its associated variational principle) for the set of physical variables, discarding anything else. Some of the variables which in Dirac's approach are related to the physical ones through constraints are quickly eliminated in F-J method together with other variables which are going to become gauge degrees of freedom. This fact explains the efficiency of F-J method: It does not produces superfluous information that is going to be discarded later on, contrary to what happens in Dirac's, where we can keep this superfluous information till the end.

Obviously, this efficiency must pay some price. Apart from the technically difficult Darboux transformation implied in the F-J reduction process, the indiscriminate elimination of variables leads to the typical difficulties that plague the reduction procedures: the general lost of covariance and in some cases even the lost of properties of locality for the reduced field theory. Let us mention that the BRST quantization methods mentioned above lie just on the "other side" of the "reductionist" approach: In any version of the BRST formalism, the set of variables (the original fields in the theory), instead of being reduced (thus loosing covariance, or locality) is enlarged with new variables (ghosts, antifields,..).

Beyond these issues of covariance or locality there is the important phenomena of quantum gauge anomalies, i.e., classical gauge symmetries that, due to the regularization procedures involved in the quantization program, do not become quantum symmetries. In such a case it is clear that one must be very careful in the classical reduction, since it cannot take quantum gauge anomalies into account, so they must show up somewhere else (for instance, in the early methods for the reduced quantization of the bosonic string, the disappearance of the Weyl anomaly was the hidden cause that the Poincare group was only correctly realized at 26 spacetime dimensions).

A covariant analysis of the reduced phase space (covariant symplectic approach) has been worked in [12]. For a comparative examination of this approach with the Hamiltonian reduction and the Peierls bracket procedure see [13]. The relation between first order Lagrangians and Dirac brackets in Darboux coordinates was worked out in [14]. On the other hand the results obtained from F-J approach has been compared with the corresponding results of Dirac method for the case of constant symplectic matrix through simple particular examples in [15] without proving the equivalence between the two approaches. An analysis from the point of view of the so called symplectic reduction procedure has been performed by the authors of [16], avoiding the difficult Darboux transformation implied in the original analysis of F-J by expanding, at each stage of the algorithm, the number of variables in the phase space. Some applications of F-J approach can be found in [17, 18, 19, 20] and the extension to the fermionic case for constant symplectic structures is accomplished in [21]. Recently the authors [22] compared the Dirac quantization program of promoting the Dirac brackets to commutators with the resulting brackets in the F-J approach using a gauge fixing

procedure.

The purpose of this paper is to analyze and explain in detail the equivalence of Dirac's method with Faddeev and Jackiw's one. It is usually said that with F-J method, the separation of constraints in first and second class, customary in Dirac's method, disappears. We will see that this is not completely true and that it is possible in the F-J method to keep track, at least partially, of the standard Dirac's classification of constraints as first class and second class.

The difference between the two stabilization procedures, that of Dirac and F-J, is that in the first case we obtain a new symplectic form (the Dirac brackets) and some first class constraints at each stage of the reduction procedure. The stabilization algorithm terminates when no new constraints appear in the theory. We end up with a gauge theory with only first class constraints in a partially reduced phase space. The final reduction is now performed by means of the gauge fixing procedure. In the F-J case, at each stage of the algorithm, we plug the constraints into the action and diagonalize (by means of a Darboux transformation) the symplectic form obtaining as a byproduct new constraints that by means of the reduction process result in a new symplectic structure that may be degenerate. The procedure continues through a new diagonalization and ends up with a non degenerate symplectic structure that actually represents the Dirac brackets in the reduced phase space.

To compare the two procedures we will work in coordinates that allow for a canonical representation of the constraint surface at each stage of the stabilization procedure. In these coordinates the Dirac brackets become diagonal at each stage of the algorithm. We assume that all the constraints that appear in the formalism are effective, i.e., its gradient on the constraint surface is different from zero. In fact, this assumption is crucial and its failure may obstruct the equivalence between Dirac's and Faddeev and Jackiw's methods.

The equivalence of Dirac's with Faddeev and Jackiw's method is proved in section 2. An example are given is section 3 and section 4 is devoted to conclusions.

2   The equivalence of Dirac's and F-J methods

We will take as starting point a canonical Hamiltonian $H_c(q,p)$ and a set of primary constraints $\phi_\mu = 0$. One can consider the phase space as the new configuration space for the canonical Lagrangian,

$$L = p_i \dot{q}^i - H_c(q,p) - \lambda^\mu \phi_\mu. \tag{2.1}$$

λ^μ are Lagrange multipliers which are taken as arbitrary functions of time –until some of them get determined by consistency requirements. The Euler-Lagrange equations for L yield the usual Hamilton-Dirac equations on the surface of primary constraints $\phi_\mu = 0$.

2.1   F-J analysis

In F-J method one eliminates as many variables as constraints ϕ_μ just by plugging these constraints –which now are holonomic– into the Lagrangian. If we call x^s the remaining set of variables, the reduced Lagrangian will take the form[1]

$$L' = a_s(x)\dot{x}^s - H(x), \tag{2.2}$$

for some functions a_s and H. At this point we can perform a Darboux transformation,[2]

$$x^s \longrightarrow Q^r, \, P_r, \, Z_a \tag{2.3}$$

such that L' takes again the form of a canonical Lagrangian:

$$L' = P_r \dot{Q}^r - H'(Q^r, P_r, Z_a) \tag{2.4}$$

Notice that in general there appear a certain number of variables, Z_a, without its canonical counterpart. This reflects the fact that the primary constraint surface can be endowed with a pre-symplectic structure, and not a symplectic one. These variables Z_a play the role of auxiliary variables. By using its equations of motion, some of them can be eliminated. Let us name Z_{a_1} the maximum set of variables that can be eliminated, and let Z_{a_2} be the rest. In other words, the equations of motion for the variables Z_a,

$$\frac{\partial H'}{\partial Z_a} = 0, \tag{2.5}$$

allow for the isolation of Z_{a_1} in terms of the rest of variables,

$$Z_{a_1} = f_{a_1}(Q^r, P_r, Z_{a_2}), \tag{2.6}$$

together with some other possible relations which are free from these Z_a variables,

$$f_{a_2}(Q^r, P_r) = 0. \tag{2.7}$$

(some of these last relations –or maybe all– may be empty). Here a_2 denotes the maximum number of independent relations of type (2.7).

Now let us proceed to a new reduction: the elimination of the variables Z_{a_1} by plugging (2.6) into (2.4). In this way we get

$$L_R = P_r \dot{Q}^r - H''(Q^r, P_r, Z_{a_2}). \tag{2.8}$$

But now the dependence of H'' with respect to Z_{a_2} cannot be more than linear –just by construction– because otherwise variables of the type Z_{a_2} could still be eliminated

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[1]Notice that the sub or super-index carries information both on the type of variable –labeled by the letter– under consideration and the range of values that it can take.

[2]For a careful analysis of the Darboux theorem for singular systems, see [23].

through the procedure above, in contradiction with the fact that we had already chosen the maximum set of variables that could be eliminated, therefore

$$L_R = P_r \dot{Q}^r - H_R(Q^r, P_r) - Z_{a_2} f_{a_2}(Q^r, P_r). \tag{2.9}$$

Note that f_{a_2} is here an equivalent representation of the constraints defined by (2.7) and that we have arrived at a reduced Lagrangian, in a reduced phase space, that have the same form as (2.1). This finishes one step –the first– in F-J algorithm. The variables Z_{a_2} play from now on the role of Lagrange multipliers. Now the procedure is repeated again and again until the disappearance of these Z-type variables from the formalism. The remaining variables will be the physical ones, associated to the true degrees of freedom of the system. Equivalently the procedure ends when we obtain a non degenerate symplectic matrix from the reduced Lagrangian. This symplectic matrix represents the Dirac brackets in the reduced phase space.

Now we are ready to start the Dirac's procedure, adapted for a transparent comparison with the F-J approach. So let us go back to (2.1). The constraints $\phi_\mu = 0$ can be classified into first and second class. Then it is possible to change to an equivalent set of constraints (i.e., a change of the basis of functions that generate the ideal of functions vanishing at the primary constraint surface),[3]

$$\phi_\mu \rightarrow \xi_\mu = M_\mu^\nu(q,p)\phi_\nu, \qquad \det M \neq 0, \tag{2.10}$$

with $\xi_\mu = \varphi_a, \chi_l, \bar{\chi}_l$, and such that

$$\{\varphi_a,\ \chi_{a'}\} = \{\varphi_a,\ \chi_l\} = \{\chi_l,\ \chi_{l'}\} = \tag{2.11}$$

$$\{\varphi_a,\ \bar{\chi}_l\} = \{\bar{\chi}_l,\ \bar{\chi}_{l'}\} = 0, \tag{2.12}$$

$$\{\chi_l,\ \bar{\chi}_{l'}\} = \delta_{l,l'}. \tag{2.13}$$

Notice that the assumption that all constraints are effective is crucial for this change of basis to exist. A canonical transformation can bring these constraints to canonical coordinates,

$$(q_i,\ p^i) \rightarrow (Q^i,\ P_i) = (Q^r,\ P_r,\ Q^a,\ P_a,\ Q^l,\ P_l,), \tag{2.14}$$

with

$$P_a = \varphi_a, \quad Q^l = \chi_l, \quad P_l = \bar{\chi}_l.$$

We can also redefine the Lagrange multipliers $\eta^\mu = M^{-1}{}_\nu^\mu \lambda^\nu$. Then the original Lagrangian can be written with the new variables,

$$L = P_i \dot{Q}^i - \bar{H}_c(Q^i, P_i) - \eta^a P_a - \eta^l P_l - \bar{\eta}^l Q^l, \tag{2.15}$$

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[3]It is worth noting that, in the case of first class constraints, the BRST formalism in the extended phase space (including ghost, antighosts ...), the transition from one set of constraints to an equivalent one (2.10) is induced by a canonical transformation in the extended phase space. This fact is at the root of the well known result that states that the BRST charge is unique up to a canonical transformation in the extended phase space. [24].

where we have discarded a total derivative $\dot{F}(Q, P)$ coming from $p_i \dot{q}^i = P_i \dot{Q}^i + \dot{F}(Q, P)$. The action induced in the reduced phase space differ weakly from the action in all phase space by the boundary term $F(Q, P)$ evaluated at the extremal points that are fixed in the variational principle. It is important to note that this term may not be invariant under the gauge transformations generated by the first class constraints[24]. For the problems that can appear with the consistence between the boundary terms and the gauge fixing, at the level of the variational principle, see[25].

With this "canonical representation of the primary constraint surface", the F-J method can be constructed as follows. Plugging the constraints $P_a = 0$, $Q^l = 0$, $P_l = 0$, into (2.15) we get

$$L' = P_r \dot{Q}^r - H'(Q^r, P_r, Q^a) \tag{2.16}$$

with $H'(Q^r, P_r, Q^a) = \bar{H}_c(Q^r, P_r, Q^a, P_a = 0, Q^l = 0, P_l = 0)$. So we have arrived directly to (2.4) and the method continues as before. Notice that we have identified the variables Z_a as Q^a, the conjugate variables to the primary first class constraints P_a.

Note that the Lagrangians (2.16) and (2.4) coincide. This fact can be explained as follows. The Darboux transformation (2.3), essentially no-canonical, shows up as a canonical transfromation in a bigger space –the phase space in which the constraint surface is embedded–. This is a particular instance of how the addition of new variables can give us more symmetry. In this case a non-canonical transformation is converted into a canonical one. Extending off the constraint surface the Darboux transformation we get a canonical transformation. Obviously the extension must be zero *on* the constraint surface and this in turn imply that the extended variables must be a combination of the constraint functions as in (2.14).

2.2 Dirac analysis

Consider now the Dirac's method. Having used the change of basis of constraints and the canonical transformation above, we can start with the canonical Hamiltonian $\bar{H}_c(Q^i, P_i)$ and the constraints $P_a = 0$, $Q^l = 0$, $P_l = 0$. The initial Dirac Hamiltonian is

$$\bar{H}_c(Q^i, P_i) + \eta^a P_a + \eta^l P_l + \bar{\eta}^l Q^l. \tag{2.17}$$

Applying the stabilization algorithm to the second class constraints Q^l, P_l we determine $\eta^l = 0$, $\bar{\eta}^l = 0$. Now a trivial implementation of the Dirac bracket allows for the elimination of the second class constraints. The new Dirac Hamiltonian, in the partially reduced phase space –for the variables Q^r, P_r, Q^a, P_a–, is

$$H'(Q^r, P_r, Q_a) + \eta^a P_a, \tag{2.18}$$

with $H'(Q^r, P_r, Q_a) = \bar{H}_c(Q^r, P_r, Q^a, P_a = 0, Q^l = 0, P_l = 0)$. We are entitled to put $P_a = 0$ within the canonical Hamiltonian because this Hamiltonian is only unambiguously defined on the surface of primary constraints. Now consider the stabilization algorithm for the primary first class constraints P_a. We get the secondary constraints

$$\dot{P}_a = \{P_a, H'\} = -\frac{\partial H'}{\partial Q^a} = 0, \tag{2.19}$$

which are nothing but (2.5). We can split, as we did before, these secondary constraints between those that allow for the elimination of the variables Q^a and the rest. Using again the splitting $a = (a_1, a_2)$, and having (2.5) and (2.6) in mind, (2.19) can be written as

$$Q^{a_1} - f_{a_1}(Q^r, P_r, Q^{a_2}) = 0, \qquad f_{a_2}(Q^r, P_r) = 0. \tag{2.20}$$

Now let us partially proceed to the second step in Dirac's method. It is well known that each step comprises the possible determination of some Lagrange multipliers and the possible introduction of new constraints. In our case it is immediate to see that the stabilization of the first set of secondary constraints in (2.20), $Q^{a_1} - f_{a_1}(Q^r, P_r, Q^{a_2}) = 0$, leads to the determination of the Lagrange multipliers η_{a_1}, whereas the stabilization of the second set, $f_{a_2}(Q^r, P_r) = 0$, leads to the potential appearance of new –tertiary– constraints. In Dirac's language, the constraints $Q^{a_1} - f_{a_1}(Q^r, P_r, Q^{a_2}) = 0$ induce the change of status of some part of the up-to-now primary first class constraints, $P_a = 0$, which will become second class. More specifically, $P_{a_1} = 0$ are the constraints that become second class. With this new set of second class constraints, $P_{a_1} = 0$, $Q^{a_1} - f_{a_1}(Q^r, P_r, Q^{a_2}) = 0$, we can take again the Dirac bracket and get rid of the canonical variables P_{a_1}, Q^{a_1}. Thus we are led to the newly reduced Dirac Hamiltonian

$$H''(Q^r, P_r, Q^{a_2}) + \eta^{a_2} P_{a_2}, \tag{2.21}$$

in a phase space with canonical variables Q^r, P_r, Q^{a_2}, P_{a_2}. H'' is H' with the substitution of Q^{a_1} using (2.20). The argument developed before tells us that the dependence of H'' with respect to Q^{a_2} is at most linear. The Dirac Hamiltonian is, therefore,

$$H_R(Q^r, P_r) + Q^{a_2} f_{a_2}(Q^r, P_r) + \eta^{a_2} P_{a_2}, \tag{2.22}$$

Notice that the last summand only have effect in the equations of motion for the variables Q^{a_2}, which become, as of now, arbitrary functions:

$$\dot{Q}^{a_2} = \eta^{a_2}, \tag{2.23}$$

This special fact allows for a further reduction that it is not usually performed in the Dirac's method: we can eliminate the last summand in (2.22), reinterpret Q^{a_2} as new Lagrange multipliers, and reduce the canonical bracket to the variables Q^r and P_r. The dynamics for this reduced set of variables will remain unchanged. So we are led to a dynamics described by

$$H_R(Q^r, P_r) + Q^{a_2} f_{a_2}(Q^r, P_r), \tag{2.24}$$

with Q^{a_2} now playing the role of Lagrange multipliers. At this point we have arrived at the same result obtained after performing the first step in F-J method, (2.9).

To summarize, one step in F-J method corresponds to one step and a half in Dirac's, plus the adoption of Dirac brackets after each step, plus the trading of the remaining Lagrange multipliers (here η^{a_2}) by a set of variables (here Q^{a_2}) that in fact are its primitives, according to (2.23). **This proves the full equivalence of both methods**.

3   Example

In particular examples it may be difficult to compare the results obtained by the two reduction procedures, that of Dirac and F-J approaches. It may happen that the reduced Lagrangian obtained from Dirac reduction coincide exactly with the reduced Lagrangian obtained from F-J reduction procedure –as in the case when we have only primary second class constraints–. Other possible instance is that the resulting Lagrangians can be related by a canonical transformation –as in the case when we choose some gauge fixing constraints in Dirac's reduction in a different way that the natural choice implicit in the F-J analysis, namely by setting to zero all the gauge degrees of freedom–. The situation may be still more involved and the two reduced Lagrangians may differ by a non-canonical transformation –as in the case when, at some stage of F-J procedure, the set of constraints (2.7) is empty–. This situation can arise because enforcing the constraints (2.6) through F-J method, does not alter the symplectic structure of the Lagrangian (2.4), while the Dirac formalism with second class constraints gives Dirac brackets that, in general, are not diagonal. In any case we have always the possibility to resort to the equivalence proof given in section 2 to analyze the explicit form of the transformation that connects the two reduced Lagrangians. The situation can be described by the following diagram

$$
\begin{array}{ccc}
L(q,p) & \xrightarrow{DIRAC-REDUCTION} & L_R(q_r,p_r) \\
\text{CANONICAL} \downarrow & & \downarrow \text{NON-CANONICAL} \\
\bar{L}(Q,P) & \xrightarrow{F-J-REDUCTION} & \bar{L}_R(Q_r,P_r)
\end{array}
$$

where the non-canonical transformation can be calculated by the restriction to the constraint surface of the canonical transformation that relates the two Lagrangians, as we show in the following example. Notice that this non-canonical transformation arises because in the F-J reduction process its necessary to perform a Darboux transformation at each level of the algorithm.

The aim of this section is to illustrate some of the ideas of the previous paragraphs by means of an example that exhibits this type of behavior. To this end let us take a dynamical system represented by the following first order Lagrangian [26].

$$L = p_1\dot{q}_1 + p_2\dot{q}_2 - H_c(q,p) - \lambda\phi_1 \tag{3.1}$$

where

$$H_c = \frac{1}{2}(p_1 - q_2)^2 - \frac{\beta}{2}(q_1 - q_2)^2, \quad \phi = p_2 - (1-\alpha)q_1, \tag{3.2}$$

with $\alpha \neq \beta$. The stabilization of ϕ leads to a secondary constraint

$$\phi_2 = \alpha(p_1 - q_2) - \beta(q_1 - q_2) = 0 \tag{3.3}$$

which upon stabilization gives

$$\alpha\beta(q_1 - q_2) - \beta(p_1 - q_2) - \gamma\lambda = 0 \tag{3.4}$$

where $\gamma \equiv \alpha^2 - \beta$. The constraints are second class if $\gamma \neq 0$ and first class when $\gamma = 0$. In the first case ($\gamma \neq 0$) the implementation of the constraints leads to the reduced Lagrangian

$$L_R = \frac{\gamma}{\alpha - \beta} \dot{q}_1 p_1 + \frac{1}{2} \frac{\beta \gamma}{(\alpha - \beta)^2} (p_1 - q_1)^2. \tag{3.5}$$

Other equivalent reductions can be implemented. Here we choose to eliminate q_2, p_2 from the constraints (3.3) and (3.4). The Dirac brackets are

$$\{q_1, q_2\}_D = \frac{\alpha}{\gamma}, \quad \{q_1, p_1\}_D = \frac{\alpha - \beta}{\gamma}, \quad \{q_2, p_2\}_D = \frac{\alpha(1 - \alpha)}{\gamma}, \tag{3.6}$$

$$\{p_1, p_2\}_D = \frac{(1 - \alpha)(\beta - \alpha)}{\gamma}, \quad \{q_1, p_2\}_D = 0, \quad \{q_2, p_1\}_D = -\frac{\beta}{\gamma}. \tag{3.7}$$

Note that the Dirac brackets in reduced phase space can be read out directly from the reduced Lagrangian (3.5).

In the second case ($\gamma = 0$) the standard reduction procedure can be implemented through a gauge fixing. The natural choice is $q_2 = 0$ and $q_1 = 0$ and the reduced Lagrangian is zero, as expected. A general gauge fixing of the form $q_1 = \chi_1(q, p), q_2 = \chi_2(q, p)$, where χ_1 does not depends on q_1 and χ_2 not depends on q_2, which we suppose that fixes the gauge without ambiguities gives, in general, a reduced Lagrangian that is a total time derivative of some function. This fact means that the two gauge fixing procedures gives first order reduced Lagrangians that differ by a canonical transformation.

Let us perform the F-J analysis for this system. In the ($\gamma \neq 0$) case the first step (implementing $\phi_1 = 0$) of F-J method gives the reduced Lagrangian

$$L = \dot{q}_1(p_1 - (1 - \alpha)q_2) - H_c(q, p), \tag{3.8}$$

up to a total time derivative. We can diagonalize the symplectic structure by means of the Darboux transformation

$$P_1 = p_1 - (1 - \alpha)q_2 \quad Q_1 = q_1 \quad Q_2 = q_2 \tag{3.9}$$

which leads to

$$L' = \dot{Q}_1 P_1 - \frac{1}{2}(P_1 - \alpha Q_2)^2 + \frac{\beta}{2}(Q_1 - Q_2)^2 \tag{3.10}$$

We note that Q_2 plays the role of a Z-type variable. From the condition (2.5) it follows that

$$Q_2 = \frac{\alpha P_1 - \beta Q_1}{\gamma}. \tag{3.11}$$

We deduce from the general analysis that this constraint is going to play the role of a second class constraint. The elimination of Q_2 from the Lagrangian (3.10) gives

$$L_R = \dot{Q}_1 P_1 + \frac{\beta}{2\gamma}(P_1 - \alpha Q_1)^2. \tag{3.12}$$

That this Lagrangian does not coincide with the corresponding Lagrangian (3.5) obtained via Dirac reduction procedure may be surprising at first sight. In fact these two Lagrangians are related via a non-canonical transformation that correspond to the restriction of the canonical transformation

$$(q_1, q_2, p_1, p_2) = (Q_1, Q_2, P_1 + (1 - \alpha)Q_2, P_2 + (1 - \alpha)Q_1), \qquad (3.13)$$

allowing for a canonical representation of the constraint surface, to the surface $P_2 = 0, Q_2 - \frac{\alpha P_1 - \beta Q_1}{\gamma} = 0$. Note that this transformation is a particular extension off the constraint surface of the Darboux transformation (3.9). The non-canonical transformation that relates this two Lagrangians is thus

$$(q_1, p_1) = (Q_1, P_1 + \frac{(1 - \alpha)(\alpha P_1 - \beta Q_1)}{\gamma}). \qquad (3.14)$$

By applying this transformation to the Dirac Lagrangian (3.5) we recover precisely the result obtained via F-J method (3.12).

The case of first class constraints, when $\gamma = 0$, can be calculated in a similar way. The first reduction gives

$$L = \dot{Q}_1 P_1 - \frac{1}{2}P_1^2 + \frac{\alpha^2}{2}Q_1^2 + Q_2(\alpha P_1 - \alpha^2 Q_1), \qquad (3.15)$$

after a proper diagonalization via the Darboux transformation (3.9). Note that the Lagrangian (3.15) has already the form (2.9) where Q_2 play, form now, the role of a Lagrange multiplier. The secondary constraint,

$$P_1 - \alpha Q_1 = 0, \qquad (3.16)$$

does not allow to obtain Q_2 as function of the rest of coordinates. As we noted, this is the condition that enable us to classify this secondary constraint as a first class constraint. The implementation of (3.16) on the Lagrangian (3.15) gives a Lagrangian which is a total time derivative. We then conclude that the reduced Lagrangian obtained via Dirac reduction and that obtained via F-J reduction are totally equivalent. They differ at most by a canonical transformation.

4   Conclusions

We have proved the full equivalence between the Dirac approach and the F-J analysis for constrained systems. While getting this equivalence we have obtained new insights into the F-J method. Let us quote some.

   First. We have identified the constraints involving the Z_{a_1} (or Q^{a_1}, see (2.6) and (2.20)) variables as the subset of secondary second class constraints that bring a subset of the primary first class constraints into second class. Here is where the old Dirac's classification of constraints into first and second class is still present in F-J method.

Second. We have said that it may be that some –or all– of the relations $f_{a_2}(P_r, Q^r) = 0$ (which are the secondary constraints that do not alter the first or second class character of the primary constraints) are empty. From the point of view of Dirac's method, this happens when the number of –independent– secondary constraints is less than the number of initial primary first class constraints. Also, from the point of view of the construction of the generators of the gauge transformations (which are made up with first class constraints in a chain involving an arbitrary function and its time derivatives), this means that there are as many chains made up with a single constraint as the difference between the two numbers just mentioned.

Third. Notice that if we start the second step in F-J method (2.9), by plugging the constraints $f_{a_2}(Q^r, P_r) = 0$ into L_R, all the information on the variables Q^{a_2} disappears. So in F-J method we will never encounter any equation for these variables. Instead, in our version of Dirac's approach, since these variables will play the role of Lagrange multipliers, either they are finally determined as functions of the rest of variables P_r, Q^r, or remain arbitrary (and hence, gauge) functions. At this point we do not know the particular fate –among these two possibilities– of each of these variables but, no matter which one will it be, we already know that these variables do not carry any relevant dynamical information and they will never account for physical degrees of freedom. Here lies the nice efficiency of F-J method we have mentioned at the introduction: getting rid of these variables as soon as possible, instead of carrying them on board until the end, as it is usually done with Dirac's method.

Fourth. From a geometrical point of view the procedure leading to the new phase space variables P_i, Q^i, beginning with the canonical representation of the constraint surface (2.10) and ending with the canonical transformation (2.14), is equivalent to extend off the constraint surface the Darboux transformation (2.3) needed in the F-J approach, as we have seen. This can be visualized as follows. We can perform a transformation of coordinates from the original phase space variables p_i, q^i to new ones x^s, ϕ_μ where $q^i(x^s), p_i(x^s)$ are the parametric equations of the constraint surface and $\phi_\mu = 0$ are the constraints. By the application of this transformation to the original canonical symplectic form we obtain a new symplectic form that in general is not diagonal. Now by means of a Darboux transformation we can diagonalize this symplectic form and obtain canonical coordinates and some variables of type Z. The new symplectic form is diagonal in a reduced phase space. Extending off the constraint surface this Darboux transformation results in the canonical transformation (2.14) used in the text.

In some cases when the matrix M_μ^ν in equation (2.10) is easy to find, the procedure presented here can be used as an alternative way to construct the Darboux transformation which is needed in the F-J approach. From this point of view the Dirac method plus the canonical transformation (2.14) that diagonalizes the Dirac brackets in a stage by stage procedure reduces to the F-J approach.

Fifth. As we have already pointed out, the F-J method is an efficient way to achieve the reduction and get the physical variables in the reduced phase-space formulation. The method is technically different but the result as compared with the standard reduced formulation is the *same*: If we introduce a gauge-fixing to eliminate the gauge

degrees of freedom in Dirac's method, we will obtain the same reduced formulation. So, as a starting point to quantization, the F-J approach presents for a general case all the problems inherent to the classical reduction of variables (posible loss of covariance, locality, or anomalies, etc.) prior to quantization.

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References

[1] P.A.M Dirac, Can. J. Math. 2, 129 (1950); Proc. R. Soc. London Ser. A 246, 326 (1958); Lectures in Quantum Mechanics (Yeshiva University, New York, 1965).

[2] J.L. Anderson and P.G. Bergmann, Phys. Rev. 83, 1018 (1951); P.G. Bergmann and I. Goldberg, Phys. Rev. 98, 531 (1955).

[3] R. Arnowitt, S. Deser and C.W. Misner, The dynamics of general relativity, in Gravitation: an introduction to current research, ed. by L. Witten, Wiley, New York (1962).

[4] A. Hanson, T. Regge, C. Teitelboim Constrained hamiltonian systems, Rome: Accad. Naz. dei Lincei (1976),

[5] M.J. Gotay, J.M. Nester and G. Hinds, J. Math. Phys. 19, 2388, (1978); M.J. Gotay and J.M. Nester, Ann. Inst. H. Poncaré A 30 (1979) 129; Ann. Inst. H. Poincaré A 32 (1980) 1.

[6] C. Batlle, J. Gomis, J.M. Pons and N. Roman-Roy, J. Math. Phys. 27, 2953, (1986).

[7] K.Kuchař, Phys. Rev D, **35**, 596 (1987) J.D.Romano and R.S.Tate, Class.Quantum Grav., **6**, 1487 (1989); K.Schleich, Class.Quantum Grav., **7**, 1529(1990); R. Loll, Phys. Rev. **D41** (1990) 3785; P.Ellicot, G. Kunstatter, and D.J. Toms, Mod. Phys. Lett. A **4**, 2397 (1989); G. Kunstatter, Class. Quantum Grav. **9**, 1469 (1992); M.S. Plyushchay and A.V. Razumov, "Dirac Versus Reduced Phase Space Quantization: Relationship of Two Methods", hep-th/9306017; C. Ordoñez and J.M. Pons, *J. Math. Phys.* **36** 1146 (1995).

[8] C. Becchi, A. Rouet and R. Stora, Phys. Lett. 52B, 344 (1974); I.V. Tyutin, gauge invariance in field theory and statistical mechanics, Lebedev preprint No. 39 (1975).

[9] L.D. Faddeev and V.N. Popov, Phys. Lett. B25, 30 (1967).

[10] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. 102 b (1981) 27; Phys. Lett. 120B (1983) 166; Phys. Rev. D28 (1983) 2567; Errata: D30 (1984) 106; Nucl. Phys. B234 (1984) 106; J. Math. Phys. 26 (1985) 172. For a review, see J. Gomis, J. París and S. Samuel, Phys.Rep. 259 (1995) 1.

[11] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692; R. Jackiw, (Constrained) quantization without tears, in: Proc. 2nd Workshop on constraints theory and quantization methods, Montepulciano, 1993 (World Scientific, Singapore, 1995).

Hwang,

[12] E. Witten, Nucl. Phys. B276, (1986) 291. C. Crnković and E. Witten in *300 years of gravitation* ed. S. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1987).

[13] G. Barnich, M. Henneaux and C. Schomblond, Phys. Rev D44 R939 (1991).

[14] Hojman, S., and Urrutia, L.F., *J. Math. Phys.* **22**, 1896 (1981).

[15] D.S. Kulshreshtha and H.J.W Müller-Kirsten, Phys. Rev. D43 (1991) 3376.

[16] J. Barcelos-Neto and C. Wotzasek, Int. J. Mod, Phys A7 (1992) 4981. C. Wotzasek Ann. Phys. 243 (1995) 76.

[17] C. Cronström and M. Noga, Nucl. Phys B428 (1994) 449.

[18] J. Barcelos-Neto and N.R.F. Braga, J. Math. Phys. 35 (1994) 3497.

[19] J.E. Paschalis and P.I. Porfyriadis, Phys. Lett. B355 (1995) 171.

[20] C. Wotzasek, Phys. Rev. D46 (1992) 2734.

[21] J. Govaerts, Int. J. Mod. Phys. A5 (1990) 3625. J. Barcelos-Neto and E.S. Cheb-Terrab, Int. J. Mod. Phys. A9 (1994) 5563.

[22] H.J.W. Müller-Kirsten and J. Zhang, Phys. Lett. A202 (1995) 241. See also: H.J.W. Müller-Kirsten and J. Zhang, Phys. Lett. A200 (1995) 243.

[23] P.J. Olver, Applications of Lie groups to differential equations, Graduate texts in mathematics 107, Springer Verlag (New York, 1986).

[24] M. Henneaux and C. Teitelboim, Quantization of gauge systems (Princeton University Press, 1992), chapter 2.

[25] M. Henneaux, C. Teitelboim and J.D. Vergara, Nucl. Phys. B387, 391 (1992).

[26] K. Sundermeyer, Constrained Dynamics, Lectures Notes in Physics 169, Springer-Verlag, (New York, 1982).